Solution of the Cubic Equation by the Cardano-Tartaglia Method

For real numbers A, B, C and D with $A \neq 0$, the following algorithm solves the cubic equation $f(x) = Ax^3 + Bx^2 + Cx + D = 0$. This equation has either three real roots (considering multiple roots, x_n as distinct linear factors $(x - x_n)$ of $f(x)$) or one real root and two complex conjugate roots. These three roots are designated as $x_0,\ x_1,\ x_2$.

Step 1: Normalize the equation by dividing through by A.

$$x^3 + (B/A)x^2 + (C/A)x + (D/A) = 0$$

Step 2: "Depress the cubic", let $x = y + \alpha$; so that

$$y^3 + 3\alpha y^2 + 3\alpha^2 y + \alpha^3 + (B/A)y^2 + 2\alpha(B/A)y + \alpha^2 B/A + (C/A)y + (C/A)\alpha + D/A = 0$$
$$y^3 + y^2(3\alpha + B/A) + y(3\alpha^2 + 2\alpha B/A + C/A) = -D/A - \alpha C/A - \alpha^2 B/A - \alpha^3$$

Let $\alpha = -\dfrac{B}{3A};\quad b = \dfrac{C}{A} - \dfrac{B^2}{3A^2};\quad c = -\dfrac{D}{A} + \dfrac{BC}{3A^2} - \dfrac{2B^3}{27A^3} \Rightarrow y^3 + by = c$

If $b = c = 0$, then $y = 0$ is a triple root and $x_0 = x_1 = x_2 = -\dfrac{B}{3A}$. So, to proceed in the algorithm, assume that not both b and c are zero.

Step 3: $y = P + Q;\quad y^3 = P^3 + Q^3 + 3PQy;\quad \Rightarrow\quad P^3 + Q^3 + y(b + 3PQ) = c$

Let $P = \dfrac{-b}{3Q} \quad\Rightarrow\quad Q^3 - c - \dfrac{b^3}{27Q^3} = 0 \quad\Rightarrow\quad Q^6 - cQ^3 - \dfrac{b^3}{27} = 0$

Step 4: Use the quadratic formula to solve for Q^3.

$$Q^3 = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{b^3}{27}};\quad P^3 = \frac{-b^3}{27\left(\dfrac{c}{2} \pm \sqrt{\dfrac{c^2}{4} + \dfrac{b^3}{27}}\right)} = \frac{-b^3\left(\dfrac{c}{2} \mp \sqrt{\dfrac{c^2}{4} + \dfrac{b^3}{27}}\right)}{27\left(\dfrac{c^2}{4} - \dfrac{c^2}{4} - \dfrac{b^3}{27}\right)} = \frac{c}{2} \mp \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}$$

So, we have the formal solution: $y = \sqrt[3]{\dfrac{c}{2} + \sqrt{\dfrac{c^2}{4} + \dfrac{b^3}{27}}} + \sqrt[3]{\dfrac{c}{2} - \sqrt{\dfrac{c^2}{4} + \dfrac{b^3}{27}}}$.

Step 5: $\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}=Re^{i\varphi}$

Case 1A: The discriminant, $\frac{c^2}{4}+\frac{b^3}{27}>0$ and $\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}>0$, then

$$R=\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}};\quad \varphi=0;\quad Q_0=R^{1/3};\quad P_0=-\frac{b}{3Q_0}=-\frac{b}{3R^{1/3}};\quad \Rightarrow y_0=R^{1/3}-\frac{b}{3R^{1/3}}\in\mathbb{R}$$

$$Q_1=R^{1/3}e^{i2\pi/3}=R^{1/3}\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right);\quad P_1=-\frac{b}{3Q_1}=-\frac{b}{3R^{1/3}}\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow y_1=\frac{1}{2}\left(-R^{1/3}+\frac{b}{3R^{1/3}}\right)+i\frac{\sqrt{3}}{2}\left(R^{1/3}+\frac{b}{3R^{1/3}}\right)\in\mathbb{C}$$

$$Q_2=R^{1/3}e^{i4\pi/3}=R^{1/3}\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right);\quad P_2=-\frac{b}{3Q_1}=-\frac{b}{3R^{1/3}}\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow y_2=\frac{1}{2}\left(-R^{1/3}+\frac{b}{3R^{1/3}}\right)-i\frac{\sqrt{3}}{2}\left(R^{1/3}+\frac{b}{3R^{1/3}}\right)\in\mathbb{C}$$

Case 1B: The discriminant $\frac{c^2}{4}+\frac{b^3}{27}>0$ and $\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}<0$, then

$$R=\left|\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right|;\quad \varphi=\pi;\quad Q_0=R^{1/3}\left[\cos(\pi/3)+i\sin(\pi/3)\right]=R^{1/3}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$P_0=-\frac{b}{3Q_0}=\frac{-b}{3R^{1/3}}\left[\cos(\pi/3)-i\sin(\pi/3)\right]=\frac{b}{3R^{1/3}}\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow y_0=\frac{1}{2}\left(R^{1/3}-\frac{b}{3R^{1/3}}\right)+i\frac{\sqrt{3}}{2}\left(R^{1/3}+\frac{b}{3R^{1/3}}\right)\in\mathbb{C}$$

$$Q_1=R^{1/3}e^{i(\pi+2\pi)/3}=-R^{1/3};\quad P_1=-\frac{b}{3Q_1}=\frac{b}{3R^{1/3}}\quad \Rightarrow y_1=-R^{1/3}+\frac{b}{3R^{1/3}}\in\mathbb{R}$$

$$Q_2=R^{1/3}e^{i(\pi+4\pi)/3}=R^{1/3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right);\quad P_2=-\frac{b}{3Q_1}=\frac{-b}{3R^{1/3}}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow y_2=\frac{1}{2}\left(R^{1/3}-\frac{b}{3R^{1/3}}\right)-i\frac{\sqrt{3}}{2}\left(R^{1/3}+\frac{b}{3R^{1/3}}\right)\in\mathbb{C}$$

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Case 1C: The discriminant $\frac{c^2}{4}+\frac{b^3}{27}>0$ and $\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}=0$, then

$$\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}=-\frac{c}{2}; \quad \Rightarrow \frac{c^2}{4}+\frac{b^3}{27}=\frac{c^2}{4}; \quad \Rightarrow b=0, \quad c=-|c|; \quad \Rightarrow y^3=c;$$

$$\Rightarrow y_0=\sqrt[3]{|c|}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right); \quad y_1=\sqrt[3]{c}; \quad y_2=\sqrt[3]{|c|}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$R=-c; \quad \varphi=\pi; \quad Q_0=|c|^{1/3}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right); \quad P_0=-\frac{0}{3Q_0}=0; \quad \Rightarrow y_0=|c|^{1/3}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\in\mathbb{C}$$

$$Q_1=|c|^{1/3}e^{i(\pi+2\pi)/3}=-|c|^{1/3}=\sqrt[3]{c}; \quad P_1=-\frac{b}{3Q_1}=0 \quad \Rightarrow y_1=-|c|^{1/3}=\sqrt[3]{c}\in\mathbb{R}$$

$$Q_2=|c|^{1/3}e^{i(\pi+4\pi)/3}=|c|^{1/3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right); \quad P_1=-\frac{b}{3Q_1}=0 \quad \Rightarrow y_2=|c|^{1/3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\in\mathbb{C}$$

So, **in all cases**, if the discriminant $\frac{c^2}{4}+\frac{b^3}{27}>0$ there is a real root and two complex conjugate roots.

Case 2A: The discriminant $\frac{c^2}{4}+\frac{b^3}{27}=0$, then $b=-3\sqrt[3]{c^2/4}$.

$$R=|c/2|; \quad \varphi=\begin{cases}0 & \text{if } c>0\\ \pi & \text{if } c<0\end{cases}; \quad Q_0=\begin{cases}(c/2)^{1/3} & \text{if } c>0\\ |c/2|^{1/3}\left(\frac{1+i\sqrt{3}}{2}\right) & \text{if } c<0\end{cases}$$

$$P_0=\frac{1}{Q_0}\sqrt[3]{c^2/4}=\begin{cases}(c/2)^{1/3} & \text{if } c>0\\ |c/2|^{1/3}\left(\frac{1-i\sqrt{3}}{2}\right) & \text{if } c<0\end{cases} \quad \Rightarrow y_0=\begin{cases}2(c/2)^{1/3}=2\sqrt[3]{c/2} & \text{if } c>0\\ |c/2|^{1/3}=-\sqrt[3]{c/2} & \text{if } c<0\end{cases}$$

$$Q_1=Re^{i(\varphi+2\pi)/3}=Q_0e^{i2\pi/3}=\begin{cases}(c/2)^{1/3}\left(\frac{-1+i\sqrt{3}}{2}\right) & \text{if } c>0\\ |c/2|^{1/3}e^{i\pi/3}e^{i2\pi/3}=-|c/2|^{1/3}=\sqrt[3]{c/2} & \text{if } c<0\end{cases}$$

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$$P_1 = -\frac{b}{3Q_1} = \frac{\sqrt[3]{c^2/4}}{Q_1} = \begin{cases} (c/2)^{1/3}\left(\frac{-1-i\sqrt{3}}{2}\right) & \text{if } c>0 \\ -|c/2|^{1/3} = \sqrt[3]{c/2} & \text{if } c<0 \end{cases} \Rightarrow y_1 = \begin{cases} -(c/2)^{1/3} = -\sqrt[3]{c/2} & \text{if } c>0 \\ -2|c/2|^{1/3} = 2\sqrt[3]{c/2} & \text{if } c<0 \end{cases}$$

$$Q_2 = Re^{i(\varphi+4\pi)/3} = Q_0 e^{i4\pi/3} = \begin{cases} (c/2)^{1/3}\left(\frac{-1-i\sqrt{3}}{2}\right) & \text{if } c>0 \\ |c/2|^{1/3}\, e^{i5\pi/3} = |c/2|^{1/3}\left(\frac{1-i\sqrt{3}}{2}\right) & \text{if } c<0 \end{cases}$$

$$P_2 = -\frac{b}{3Q_2} = \frac{\sqrt[3]{c^2/4}}{Q_2} = \begin{cases} (c/2)^{1/3}\left(\frac{-1+i\sqrt{3}}{2}\right) & \text{if } c>0 \\ |c/2|^{1/3}\left(\frac{1+i\sqrt{3}}{2}\right) & \text{if } c<0 \end{cases} \Rightarrow y_2 = \begin{cases} -(c/2)^{1/3} & \text{if } c>0 \\ |c/2|^{1/3} & \text{if } c<0 \end{cases} = -\sqrt[3]{c/2}$$

Thus, the three roots to $y^3+by-c=0$ with $b=-3\sqrt[3]{c^2/4}$ are $2\sqrt[3]{c/2}$ and a double root of $-\sqrt[3]{c/2}$.

That these are indeed the solutions can be verified by the following argument. For any value of c one of the solutions should be $y=\sqrt[3]{\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}}+\sqrt[3]{\frac{c}{2}-\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}}=2\sqrt[3]{c/2}$. Performing synthetic division on $y^3+0y^2-3\sqrt[3]{c^2/4}\,y-c$ at $y=2\sqrt[3]{c/2}$ gives

$$\begin{array}{r|cccc} & 1 & 0 & -3\sqrt[3]{c^2/4} & -c \\ \hline 2\sqrt[3]{c/2} & 1 & 2\sqrt[3]{c/2} & \sqrt[3]{c^2/4} & 0 \end{array}.$$

So, $y^3-3\sqrt[3]{c^2/4}\,y-c=\left(y-2\sqrt[3]{c/2}\right)\left(y^2+2\sqrt[3]{c/2}\,y+\sqrt[3]{c^2/4}\right)$ and from the quadratic formula, $y^2+2\sqrt[3]{c/2}\,y+\sqrt[3]{c^2/4}=0$ has a double root at $y=-\sqrt[3]{c/2}\pm\sqrt{\sqrt[3]{c^2/4}-\sqrt[3]{c^2/4}}=-\sqrt[3]{c/2}$. In fact, $y^3+by-c=0$ has a double root if and only if the discriminant, $\frac{c^2}{4}+\frac{b^3}{27}=0$. Suppose that β is a double root and γ is the remaining root. Since complex roots come in conjugate pairs, both β and γ are real.

$$y^3+by-c=\left(y^2-2\beta y+\beta^2\right)(y-\gamma)=y^3-y^2(\gamma+2\beta)+y\beta(\beta+2\gamma)-\gamma\beta^2$$

So, $\gamma = -2\beta$, $b = -3\beta^2$ and $c = -2\beta^3$. Thus, $\frac{c^2}{4} + \frac{b^3}{27} = \beta^6 - \beta^6 = 0$.

Case 2B: The discriminant $\frac{c^2}{4} + \frac{b^3}{27} < 0$, then $\frac{c}{2} + i\sqrt{-\frac{c^2}{4} - \frac{b^3}{27}} = Re^{i\varphi}$, so

$$R^2 = \frac{c^2}{4} - \left(\frac{c^2}{4} + \frac{b^3}{27}\right) = \frac{-b^3}{27}; \quad \cos(\varphi) = \frac{c}{2R} = \frac{3c\sqrt{-3b}}{b^2}; \quad \sin(\varphi) = \frac{\sqrt{-\frac{c^2}{4} - \frac{b^3}{27}}}{R} = \sqrt{1 + \frac{27c^2}{4b^3}};$$

$$\varphi = \cos^{-1}\left(\frac{3c\sqrt{-3b}}{b^2}\right); \quad Q_0 = \sqrt{\frac{-b}{3}}\left[\cos(\varphi/3) + i\sin(\varphi/3)\right]$$

$$P_0 = -\frac{b}{3Q_0} = \sqrt{\frac{-b}{3}}\left[\cos(\varphi/3) - i\sin(\varphi/3)\right] \quad \Rightarrow y_0 = 2\sqrt{\frac{-b}{3}}\cos(\varphi/3) \in \mathbb{R}$$

$$Q_1 = Q_0 e^{i2\pi/3} = \sqrt{\frac{-b}{3}}\left[\cos\left(\frac{\varphi + 2\pi}{3}\right) + i\sin\left(\frac{\varphi + 2\pi}{3}\right)\right]$$

$$P_1 = -\frac{b}{3Q_1} = \sqrt{\frac{-b}{3}}\left[\cos\left(\frac{\varphi + 2\pi}{3}\right) - i\sin\left(\frac{\varphi + 2\pi}{3}\right)\right]; \quad \Rightarrow y_1 = 2\sqrt{\frac{-b}{3}}\cos\left(\frac{\varphi + 2\pi}{3}\right) \in \mathbb{R}$$

$$Q_2 = Q_0 e^{i4\pi/3} = \sqrt{\frac{-b}{3}}\left[\cos\left(\frac{\varphi + 4\pi}{3}\right) + i\sin\left(\frac{\varphi + 4\pi}{3}\right)\right]$$

$$P_2 = -\frac{b}{3Q_1} = \sqrt{\frac{-b}{3}}\left[\cos\left(\frac{\varphi + 4\pi}{3}\right) - i\sin\left(\frac{\varphi + 4\pi}{3}\right)\right] \quad \Rightarrow y_2 = 2\sqrt{\frac{-b}{3}}\cos\left(\frac{\varphi + 4\pi}{3}\right) \in \mathbb{R}$$

Thus, **in all cases**, when the discriminant $\frac{c^2}{4} + \frac{b^3}{27} \le 0$ there are three real roots.

Step 6: $x_0 = y_0 - \frac{B}{3A}; \quad x_1 = y_1 - \frac{B}{3A}; \quad x_2 = y_2 - \frac{B}{3A}$

An Excel spreadsheet that utilizes this algorithm to solve cubic equations can be accessed at the following site.

http://my.execpc.com/~aplehnen/Cubic_Solution.xls