The “Two Trig” Lemmas

The following two types of trigonometric integrals occur often enough to justify gathering the results under the heading of “trig lemmas”.

Trig Lemma 1: For *j*, *k*, *m* and *n*, whole numbers (non negative integers),

$$\int_0^{2\pi} \sin^{2j+1}(\theta)\cos^m(\theta)\,d\theta = \int_0^{2\pi} \sin^n(\theta)\cos^{2k+1}(\theta)\,d\theta = 0.$$

Proof:

$$\int_0^{2\pi} \sin^{2j+1}(\theta)\cos^m(\theta)\,d\theta = \int_0^{2\pi} \left(\sin^2(\theta)\right)^j \cos^m(\theta)\sin(\theta)\,d\theta = \int_0^{2\pi} \left(1-\cos^2(\theta)\right)^j \cos^m(\theta)\sin(\theta)\,d\theta$$

Now let $u = \cos(\theta)$, then $du = -\sin(\theta)\,d\theta$ and the result follows since

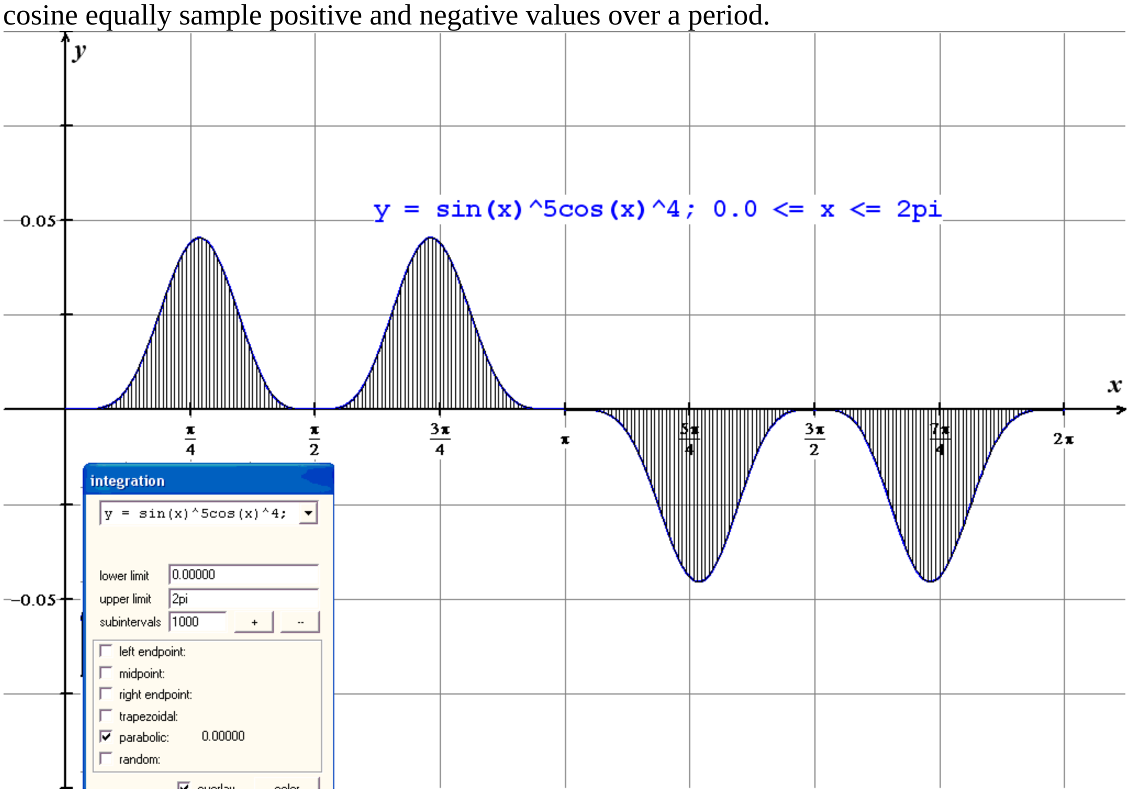
$\int_0^{2\pi} \sin^{2j+1}(\theta)\cos^m(\theta)\,d\theta = -\int_{\cos(0)}^{\cos(2\pi)} \left(1-u^2\right)^j u^m\,du = -\int_1^1 \left(1-u^2\right)^j u^m\,du = 0$. Similarly,

$$\int_0^{2\pi} \sin^n(\theta)\cos^{2k+1}(\theta)\,d\theta = \int_0^{2\pi} \left(\cos^2(\theta)\right)^k \sin^n(\theta)\cos(\theta)\,d\theta = \int_0^{2\pi} \left(1-\sin^2(\theta)\right)^k \sin^n(\theta)\cos(\theta)\,d\theta$$

Let $u = \sin(\theta)$, then $du = \cos(\theta)\,d\theta$ and again the result follows since

$$\int_0^{2\pi} \sin^n(\theta)\cos^{2k+1}(\theta)\,d\theta = \int_{\sin(0)}^{\sin(2\pi)} \left(1-u^2\right)^k u^n\,du = \int_0^0 \left(1-u^2\right)^k u^n\,du = 0.$$

It is **important** that the domain of integration be a period or “**full cycle**” of $[0, 2\pi]$ since the result is due to exact cancellation of positive area with negative area and fact that positive odd powers of sine and cosine equally sample positive and negative values over a period.



Trig Lemma 2: For any integer *k* $\int_0^{k(\pi/2)} \sin^2(\theta)d\theta = \int_0^{k(\pi/2)} \cos^2(\theta)d\theta = \frac{k\pi}{4} = \frac{1}{2}\left(\frac{k\pi}{2}\right)$**.**

In words: the integral of either sine squared or cosine squared over an integer multiple of quarter periods is half the width of the interval of integration.

Proof 1: Integration by parts

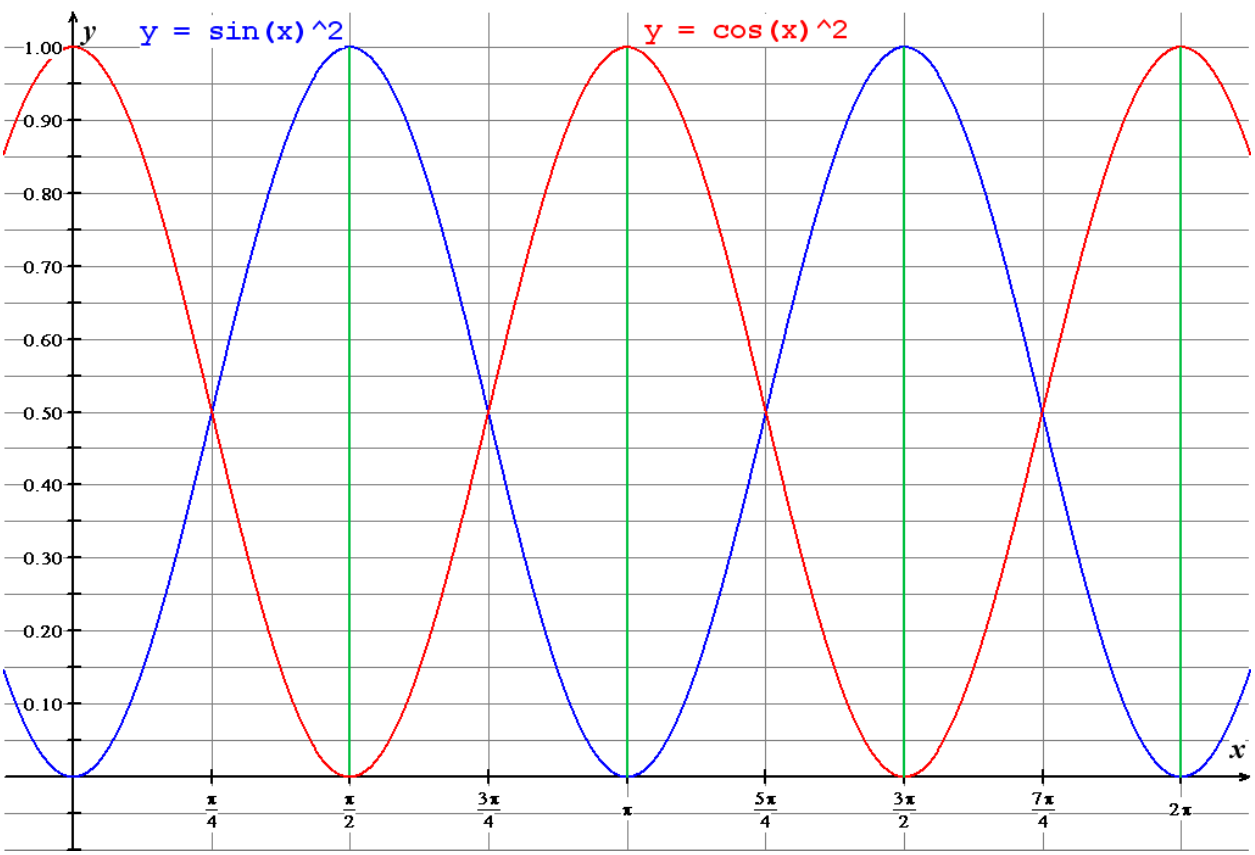
$$\int_0^{k(\pi/2)} \sin^2(\theta)d\theta = \int_0^{k(\pi/2)} \sin(\theta)\left(\sin(\theta)d\theta\right) = -\cos(\theta)\sin(\theta)\Big|_0^{k(\pi/2)} + \int_0^{k(\pi/2)} \cos^2(\theta)d\theta$$

$$= -\cos\left(k(\pi/2)\right)\sin\left(k(\pi/2)\right) + \int_0^{k(\pi/2)} \cos^2(\theta)d\theta$$

Now k is either even or odd and sine vanishes at multiples of π while cosine vanishes at odd multiples of $\frac{\pi}{2}$. If k is even, $-\cos\left(k(\pi/2)\right)\sin\left(k(\pi/2)\right) = -\cos\left(k(\pi/2)\right)\sin\left(\frac{k}{2}\pi\right) = 0$, while if k is odd, $-\cos\left(k(\pi/2)\right)\sin\left(k(\pi/2)\right) = 0$. Hence for every integer k,

$$\int_0^{k(\pi/2)} \sin^2(\theta)d\theta = \int_0^{k(\pi/2)} \cos^2(\theta)d\theta\,.$$

Interpreting the definite integral as the area under the curve also leads to this identity as the figure below illustrates.



Hence,

$$\int_0^{k(\pi/2)} \sin^2(\theta)d\theta = \frac{1}{2}\left[\int_0^{k(\pi/2)} \sin^2(\theta)d\theta + \int_0^{k(\pi/2)} \sin^2(\theta)d\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{k(\pi/2)} \sin^2(\theta)d\theta + \int_0^{k(\pi/2)} \cos^2(\theta)d\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{k(\pi/2)} \left(\sin^2(\theta) + \cos^2(\theta)\right)d\theta = \frac{1}{2}\int_0^{k(\pi/2)} 1d\theta\right] = \frac{1}{2}\left(\frac{k\pi}{2}\right) = \frac{k\pi}{4}$$

So for any integer k, $\int_0^{k(\pi/2)} \sin^2(\theta)d\theta = \int_0^{k(\pi/2)} \cos^2(\theta)d\theta = \frac{k\pi}{4} = \frac{1}{2}\left(\frac{k\pi}{2}\right)$.

Proof 2: Using the double angle formula

$$\cos(2\theta) = \cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) = \cos^2(\theta) - \left(1-\cos^2(\theta)\right) = 2\cos^2(\theta) - 1$$

$$= 1-\sin^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$$

So we have the “half angle” identities:

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

Thus,

$$\int_0^{k(\pi/2)} \sin^2(\theta)d\theta = \frac{1}{2}\int_0^{k(\pi/2)} \left(1-\cos(2\theta)\right)d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{k(\pi/2)}$$

$$= \frac{1}{2}\left(\frac{k\pi}{2} - \frac{1}{2}\sin(k\pi)\right) = \frac{1}{2}\left(\frac{k\pi}{2}\right)$$

and

$$\int_0^{k(\pi/2)} \cos^2(\theta)d\theta = \frac{1}{2}\int_0^{k(\pi/2)} \left(1+\cos(2\theta)\right)d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{k(\pi/2)}$$

$$= \frac{1}{2}\left(\frac{k\pi}{2} + \frac{1}{2}\sin(k\pi)\right) = \frac{1}{2}\left(\frac{k\pi}{2}\right)$$